Lecture 11 - הרצאה 11

Continuing Mermin-Wagner thm.

Consider the XY model as a field of angles $(\theta_v)_{v\in\Lambda} \subseteq \mathbb{R}/2\pi\mathbb{Z}$

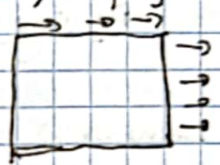
Density of XY model:

$$\rho(\theta) = \frac{1}{Z}\prod_{\substack{u\sim v\\ u,v\in\Lambda}} e^{\beta\cos(\overbrace{\theta_u-\theta_v}^{\text{angle}})}\qquad \text{angle} \in [-\pi,\pi]$$

With respect to $d\theta$ which is the product Lebesgue measure.

We restrict to configurations with $\theta_v = 0$ on

call it $\partial_o\Lambda$ → bdy of Λ, and take $\Lambda = \Lambda_L = \{-L,\dots,L\}^2$.



Mermin-Wagner thm.: $\forall\beta>0$,

$$\lim_{L\to\infty} \|\mathbb{E}\, e^{i\theta_0}\| = 0$$

We emphasize that this is a two-dim. phenom.

Proof:

Lemma 1: For any $\tau:\Lambda\to\mathbb{R}$, $\tau(\partial_o\Lambda)=0$ and any $\theta:\Lambda\to\mathbb{R}/2\pi\mathbb{Z}$,

$$\sqrt{\rho(\theta+\tau)\cdot\rho(\theta-\tau)} \geq \rho(\theta)\cdot e^{-C_1\underbrace{\sum_{u\sim v}(\tau_u-\tau_v)^2}_{\text{a "cost" of the perturbation } \tau}}$$

for a universal $C_1>0$

Lemma 2: $\exists\,\tau:\Lambda\to\mathbb{R}$, $\tau(\partial_o\Lambda)=0$, $\tau(0)=\pi$ with $\sum_{u\sim v}(\tau_u-\tau_v)^2 \leq \frac{C}{\log L}$ for a universal $C_1>0$.

We proved Lemma 2 last time.

Proof of Lemma 1:

By the explicit formula:

$$\sqrt{p(\theta+\tau)\,p(\theta-\tau)} = \frac{1}{Z}\prod_{u\sim v} e^{\frac{1}{2}\beta\left(\cos(\theta_u-\theta_v+\tau_u-\tau_v)+\cos(\theta_u-\theta_v-(\tau_u-\tau_v))\right)} = (*)$$

Taylor expansion:

$$\cos(a+\delta) \geq \cos(a) - \sin(a)\cdot\delta - C_1\cdot\delta^2$$

For any a, δ.

$$(*) \geq \frac{1}{Z}\prod_{u\sim v} e^{\beta\left(\cos(\theta_u-\theta_v) - C_1(\tau_u-\tau_v)^2\right)}$$

The first order ~~term~~ vanishes due to the geometric average!

Proving Lemma 1.

Lemma 2 is a two-dim. fact.

Lemma 1 is general.

Lemma 3: Denote $C_\tau = e^{-C_1\beta\sum_{u\sim v}(\tau_u-\tau_v)^2}$ the factor in Lemma 1.

For each event A we have:

$$\sqrt{\mathbb{P}(\theta\in A+\tau)\cdot\mathbb{P}(\theta\in A-\tau)} \geq \mathbb{P}(\theta\in A)\cdot C_\tau.$$

$$(A+\tau = \{\theta+\tau : \theta\in A\}).$$

proof: Let $I_A = \int_A \sqrt{p(\theta+\tau)\,p(\theta-\tau)}\,d\theta$

From Lemma 1,

$$I_A \geq \int_A p(\theta)\,d\theta\cdot C_\tau = \mathbb{P}(\theta\in A)\cdot C_\tau$$

By the Cauchy-Schwartz ineq.

$$I_A \leq \sqrt{\int_A p(\theta+\tau)\,d\theta\int_A p(\theta-\tau)\,d\theta} = \sqrt{\mathbb{P}(\theta\in A-\tau)\cdot\mathbb{P}(\theta\in A+\tau)}$$

change of variables
$\theta\mapsto\theta+\tau$
or
$\theta\mapsto\theta-\tau$
Jacobian $= 1$.

Lemma 4:

For any $x \in \mathbb{R}/2\pi\mathbb{Z}$ and $0 \le t \le \pi$,

$$\sqrt{\rho_0(x+t) \cdot \rho_0(x-t)} \ge \rho_0(x) \cdot e^{-\frac{c\delta}{\log L}}$$

Where ρ_0 is the density of Θ_0 with respect to the Lebesgue measure.

The lemma easily implies that ρ_0 is almost constant.

spin at the origin will have an almost uniform density.

Indeed, ρ_0 is cts. and if x_0 is such that

$$\rho_0(x_0) = \max_x \rho_0(x)$$

then we conclude for all t,

$$\sqrt{\rho_0(x_0+t)\,\rho_0(x_0)} \underset{\text{def. of } x_0}{\ge} \sqrt{\rho_0(x_0+t)\,\rho_0(x_0-t)} \ge \rho_0(x_0) \cdot e^{-\frac{c}{\log L}}$$

$$\Rightarrow \rho_0(x+t) \ge \rho_0(x_0) \cdot e^{-\frac{2c}{\log L}}$$

We conclude that $\| \mathbb{E}\, e^{i\Theta_0} \| \le \frac{c\delta}{\log L}$

In fact, it is know that $\| \mathbb{E}\, e^{i\Theta_0} \| \le \frac{C(\beta)}{L^{c(\beta)}}$

and maybe we'll see an exercise in the homework

Proof of lemma 4: Let τ be the function from lemma 2. Let $I \subseteq \mathbb{R}/2\pi\mathbb{Z}$ be an interval! $\underset{I}{\vdash\!\!-\!\!\dashv}$

Let $A = \{\Theta_0 \in I\}$.

By lemma 3., for any $0 \le s \le 1$,

$$\sqrt{\underbrace{\mathbb{P}(\Theta \in A + s\cdot\tau)}_{= \mathbb{P}(\Theta_0 \in I + s\pi)} \cdot \underbrace{\mathbb{P}(\Theta \in A - s\cdot\tau)}_{= \mathbb{P}(\Theta_0 \in I - s\pi)}} \ge \underbrace{\mathbb{P}(\Theta \in A)}_{\mathbb{P}(\Theta \in A)} e^{-\frac{C\beta s^2}{\log L}} \ge \mathbb{P}(\Theta \in A)\, e^{-\frac{C\delta}{\log L}}$$

Divide by the length of I and shrink I to the point x to get the result. (with $t = s\cdot\pi$)

Long-range order in the spin $O(n)$ model.

In dimensions $d \geq 3$

$n \geq 2$: Spin $O(n)$ model means that spins take value in S^{n-1}.

$$H(\sigma) = -2\sum_{u\sim v} \underbrace{\sigma_u \cdot \sigma_v}_{\text{inner product in } \mathbb{R}^n.}$$

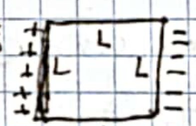
$$\sigma_u = (\sigma_u^1, \text{---}, \sigma_u^n), \quad \sigma_u \cdot \sigma_v = \sum_{j=1}^{n} \sigma_u^j \cdot \sigma_v^j$$

Roughly, we will show that the correlations $\mathbb{E}[\sigma_u \cdot \sigma_v]$ do not decay to zero as $\|u-v\| \to \infty$ in dimensions $d \geq 3$, at low temp. (β large).

In contrast to the Mermin-Wagner thm. which holds when $d=2$.

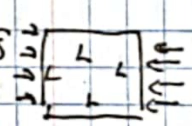
Remark: We already saw this for the Ising model with a Peierls argument.

In Ising: the "energetic cost" of



is L^{d-1}!

In the spin $O(n)$ model, $n \geq 2$, the "cost" of



is L^{d-2}.

The lower cost already raises a suspicion that there will be no long-range order in two dim. [confirmed by the Mermin-Wagner thm]. and also implies that one will not be able to use any sort of Peierls argument when $d \geq 3$.

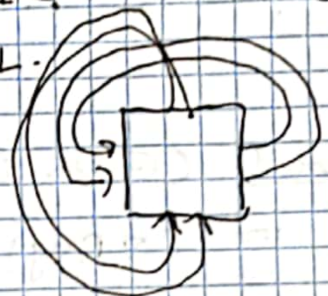
We lack methods for proving long-range order in continuous-symmetry models. We'll show today the infra-red bound which is the best and only method for the spin $O(n)$ model.

save for a 500-page work of Balaban on a related model.

Open: Prove long-range order in the Quantum Heisenberg Ferromagnet in $d \geq 3$, low temp.

The proof requires a lot of symmetries in the domain so we'll work on the torus $\Lambda = \{-\frac{L}{2}+1, \dots, \frac{L}{2}\}^d$

a discrete torus, with even L.



Configuration space is: $\Omega_\Lambda = \{\sigma : \Lambda \to S^{n-1}\}$

Hamiltonian.

$$H(\sigma) = -2 \sum_{\substack{u \sim v \\ u,v \in \Lambda}} \sigma_u \cdot \sigma_v = \underset{\substack{\uparrow \\ \sigma_u, \sigma_v \in S^{n-1}}}{\sum_{u \sim v}} \|\sigma_u - \sigma_v\|_2^2 - 2d|\Lambda|$$

prob. density is $\overset{\text{proportional to}}{\sim} e^{-\beta H(\sigma)}$ w.r.t. $d\sigma$ which is product Lebesgue measure.

<u>Two point function:</u> $\gamma(x) = \mathbb{E}(\sigma_0^1 \cdot \sigma_x^1)$ (first comp. of σ_0 and σ_x)

We'll show that $\gamma(x) \not\to 0$ as $x \to \infty$

<u>Dual torus:</u> $\Lambda^* = \frac{2\pi}{L}\Lambda = \frac{2\pi}{L}\{-\frac{L}{2}+1, \dots, \frac{L}{2}\}^d$

<u>Eigenvalues of Δ:</u> $\mathcal{E}(k) = 2\sum_{i=1}^{d}(1-\cos(k_i))$ for $k \in \Lambda^*$

near $k=0$, $\mathcal{E}(k) \approx |k|_2^2$.

<u>Thm.</u> (Fröhlich – Simon – Spencer 1976):

$$\frac{1}{|\Lambda|}\sum_{u \in \Lambda} \gamma(u) \geq \frac{1}{n} - \frac{1}{\beta} \cdot \underbrace{\frac{1}{|\Lambda|}\sum_{k \in \Lambda^* \setminus \{0\}} \frac{1}{\mathcal{E}(k)}}$$

n is the number of components

as $L \to \infty$ converges to $C\int \frac{dk}{\mathcal{E}(k)} < \infty$, in $d \geq 3$.

$\Rightarrow$ in $d \geq 3$, when β is large, $\frac{1}{|\Lambda|}\sum_{u \in \Lambda} \gamma(u) \geq C(n) > 0$ ← Long range order

For a constant $C(n)$ depending only on n.

proof of thm.

Laplacian: Δ is a $\Lambda \times \Lambda$ matrix with

$$\Delta_{u,v} = \begin{cases} 1 & u \sim v \\ -2d & u = v \\ 0 & \text{otherwise} \end{cases}$$

Fact 1: (Green identity / integration by parts)

$$= (\nabla f, \nabla g) = (f, -\Delta g)$$

For each $f, g : \Lambda \to \mathbb{R}$,

$$\sum_{\substack{u \sim v \\ u,v \in \Lambda}} (f_u - f_v)(g_u - g_v) = -\sum_{v \in \Lambda} f_u \underbrace{(\Delta g)_u}_{\sum_{v: v \sim u} (g_v - g_u)}$$

Fact 2: For $f : \Lambda \to \mathbb{R}$, $f_k(u) = \cos(k \cdot u)$, $k \in \Lambda^*$.

$$-\Delta f_k = \mathcal{E}(k) \cdot f_k \quad \left(\begin{matrix} \text{i.e. } f_k \text{ is an eigenfunction} \\ \text{with an eigenvalue } \mathcal{E}(k) \end{matrix} \right).$$

First part of proof: Deducing long-range order from "Gaussian domination".

For $f : \Lambda \to \mathbb{R}$, define $Z(f) = \int_{\Omega_\Lambda} e^{-\beta \sum_{u \sim v} \|(\sigma_u + f_u e_1 - (\sigma_v + f_v e_1)\|_2^2} d\sigma$. $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

a shifted partition function.

For $f = 0$, get the partition function of the spin $O(n)$ model (multiplied by a constant).

GD) Gaussian domination: For every $f : \Lambda \to \mathbb{R}$, $Z(f) \le Z(0)$.

Equivalently, $\mathbb{E}\left(e^{-2\beta \sum_{u \in \Lambda} \sigma_u^1 \cdot (-\Delta f)_u}\right) \le e^{+\beta \sum_{u \sim v} (f_u - f_v)^2}$

GD means that correlations of the kind are dominated by the same correlation for the Gaussian Free Field

$$\left(\begin{matrix} \sigma : \Lambda \to \mathbb{R}^n \\ H(\sigma) = \sum_{u \sim v} \|\sigma_u - \sigma_v\|_2^2 \end{matrix} \right).$$

The equivalence:

$$\Delta = \frac{Z(\beta)}{Z(0)} = \frac{\int_{\Omega_\Lambda} e^{-\beta \sum_{u\sim v} \|\sigma_u + f_u e_1 - (\sigma_v + f_v e_1)\|_2^2}}{\int_{\Omega_\Lambda} e^{-\beta \sum_{u\sim v} \|\sigma_u - \sigma_v\|_2^2}}$$

Where for $g:\Lambda \to \mathbb{R}^n$,
∇g: edges $\to \mathbb{R}^n$
$\nabla g(u,v) = g(u) - g(v)$

Note, $\sum_{u\sim v} \|\sigma_u + f_u e_1 - (\sigma_v + f_v e_1)\|_2^2 = (\nabla(\sigma + f e_1), \nabla(\sigma + f e_1))$

$$= (\nabla\sigma, \nabla\sigma) + 2(\nabla\sigma, \nabla f e_1) + (\nabla f e_1, \nabla f e_1)$$

$(\nabla\sigma, \nabla\sigma) = \sum_{u\sim v} \|\sigma_u - \sigma_v\|_2^2$; $(\nabla\sigma, \nabla f e_1) \overset{\text{Fact 1}}{=} (\sigma_1, -\Delta f e_1) = \sum_{u\in\Lambda} \sigma_u^1 (-\Delta f)_u$; $(\nabla f e_1, \nabla f e_1) = \sum_{u\sim v} (f_u - f_v)^2$

$$\Rightarrow \frac{Z(\beta)}{Z(0)} = \underbrace{\frac{1}{Z(0)} \cdot \int_{\Omega_\Lambda} e^{-\beta \sum_{u\sim v} \|\sigma_u - \sigma_v\|_2^2}}_{\text{Average with respect to the spin O(n) measure}} \cdot e^{-2\beta \sum_u \sigma_u^1 (-\Delta f)_u} \cdot e^{-\beta \sum_{u\sim v} (f_u - f_v)^2} \, d\sigma.$$

$$= \mathbb{E}\left(e^{-2\beta \sum_u \sigma_u^1 (-\Delta f)_u - \beta \sum_{u\sim v} (f_u - f_v)^2}\right).$$

GD $\longrightarrow$ bound on variances:

For each $f:\Lambda \to \mathbb{R}$ and each $\varepsilon > 0$

$$\mathbb{E}\left(e^{-2\beta \sum_{u\in\Lambda} \sigma_u^1 (-\Delta \varepsilon f)_u}\right) \le e^{\beta \sum_{u\sim v} (\varepsilon f_u - \varepsilon f_v)^2}$$

Taking a Taylor expansion as $\varepsilon \to 0$,

$$1 - \varepsilon \underbrace{\mathbb{E}\left(2\beta \sum_{u\in\Lambda} \sigma_u^1 (-\Delta f)_u\right)}_{=0, \text{ by symmetry of } \sigma} + \frac{\varepsilon^2}{2} \mathbb{E}\left[\left(-2\beta \sum_{u\in\Lambda} \sigma_u^1 (-\Delta f)_u\right)^2\right] + o(\varepsilon^2)$$

$$\le 1 + \beta \varepsilon^2 \sum_{u\sim v} (f_u - f_v)^2 + o(\varepsilon^2)$$

$$\Rightarrow 2\beta^2 \operatorname{Var}\left(\sum_{u\in\Lambda} \sigma_u^1 (-\Delta f)_u\right) \le \beta \sum_{u\sim v} (f_u - f_v)^2.$$

Now take $f = f_k$ $\left(f_k(u) = \cos(k\cdot u), k \in \Lambda^*\right)$ and see what we get.

By fact 2, $-\Delta f_k = \varepsilon(k) f_k$.

$$2\beta\varepsilon(k)\,\mathrm{Var}\Big(\sum_{u\in\Lambda}\sigma_u^1 f_{k,u}\Big)\le \beta\,\varepsilon(k)\|f_k\|_2^2$$

$$\Rightarrow \underbrace{\mathrm{Var}\Big(\sum_{u\in\Lambda}\sigma_u^1 f_{k,u}\Big)}\le \frac{1}{2\beta\varepsilon(k)}\|f_k\|_2^2.$$

$$=\mathbb{E}\Big[\Big(\sum_{u\in\Lambda}\sigma_u^1 f_{k,u}\Big)^2\Big]=\sum_{u,v\in\Lambda} f_{k,u} f_{k,v}\underbrace{\mathbb{E}[\sigma_u^1\sigma_v^1]}_{\gamma(u-v)}$$

$$=\sum_{u,v\in\Lambda}\cos(k\cdot u)\cos(k\cdot v)\cdot\gamma(u-v)=\sum_{u\in\Lambda}\cos(k\cdot u)\cdot\overbrace{\sum_{w\in\Lambda}\cos(k(u-w))\gamma(w)}^{\mathrm{Re}\big[e^{iku}\cdot\underbrace{\sum e^{ikw}\gamma(w)}_{=:\hat\gamma(k)}\big]}$$

$$=\hat\gamma(k)\sum_{u\in\Lambda}\cos(k\cdot u)^2=$$

$$=\hat\gamma(k)\,\|f_k\|_2^2$$

real since γ is symmetric.

In conclusion, $\hat\gamma(k)\le\dfrac{1}{2\beta\varepsilon(k)}$ (the <u>infra-red bound</u>)

and $\hat\gamma$ is the Fourier transform on Λ of γ.

$$\hat\gamma(k)=\sum_{w\in\Lambda}e^{ik\cdot w}\cdot\gamma(w).$$

Fourier inversion formula:

$$\gamma(u)=\frac{1}{|\Lambda|}\sum_{k\in\Lambda^*}e^{iku}\hat\gamma(k)$$

~~Use Parseval's~~

Note that

$$\gamma(0)=\mathbb{E}((\sigma_0^1)^2)=\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}((\sigma_0^j)^2)=\frac{1}{n}.$$

$$\frac{1}{|\Lambda|}\sum_{k\in\Lambda^*}e^{ik\cdot 0}\cdot\hat\gamma(k)=\frac{1}{|\Lambda|}\cdot\hat\gamma(0)+\frac{1}{|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\hat\gamma(k)$$

$$\underset{\text{infra-red bound}}{\le}\frac{1}{|\Lambda|}\hat\gamma(0)+\frac{1}{|\Lambda|\beta}\sum_{k\in\Lambda^*\setminus\{0\}}\frac{1}{\varepsilon(k)}$$

$$\underbrace{\frac{1}{|\Lambda|}\hat\gamma(0)}_{=\frac{1}{|\Lambda|}\sum_{k\in\Lambda}\gamma(k)}.$$

proving long range order modulo Gaussian Domination